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Abstract

This paper shows how de Finetti's book-making principle, commonly used to justify additive subjective probabilities, can be modified to agree with some nonexpected utility models. More precisely, a new foundation of the rank-dependent models is presented that is based on a comonotonic extension of the book-making principle. The extension excludes book-making only if all gambles considered induce a same rank-ordering of the states of nature through favorableness of their associated outcomes, and allows for nonadditive probabilities. Typical features of rank-dependence, hedging, ambiguity aversion, and pessimism and optimism, can be accommodated.

Keywords: Book-Making, Comonotonic, Choquet expected utility, ambiguity aversion, ordered vector space

JEL Classification Number: D81, C60

1 Introduction

De Finetti's book-making principle entails that a gambler should not endorse preferences that can be linearly combined into a sure loss. A surprising implication is that all uncertainties have to be expressed in terms of additive probabilities, possibly subjective (de Finetti [10, 11, 12]). The principle has, since its discovery, served as a justification of Bayesianism. The main restriction of the book-making principle is that it requires outcomes to be expressed in utils, in other words, utility must be linear. This requirement is reasonable for small stakes. Linear combinations of gambles naturally arise in financial markets, where assets can be bought and sold at fixed rates. The book-making principle then amounts to a no-arbitrage requirement which is commonly accepted as normative in finance ([29], [44].

There are many descriptive reasons, and according to some authors also normative reasons, for deviations from Bayesianism. This insight has resulted from the Allais [3] and Ellsberg [15] paradoxes and has led to a rich literature ([7] [39], [41]. The most popular models today are the rank-dependent models (Quiggin [31], Schmeidler [38], Tversky & Kahneman [43], Yaari [49], They allow for nonlinear sensitivity towards uncertainty, modeled through nonadditive measures (capacities). Decision weights of events depend on how favorable the outcomes of the events are in comparison to the alternative outcomes of the gamble under consideration (rank-dependence). Basic rationality requirements such as transitivity and monotonicity are maintained but many other deviations from Bayesianism can be accommodated. Examples are pessimism (aversion to uncertainty and convex capacities), optimism (concave capacities), and insufficient sensitivity towards uncertainty (inverse-S capacities, overweighting unlikely events and underweighting likely events [43]).

In financial portfolios, investing in negatively correlated assets (hedging) is desirable. This phenomenon can be modeled by pessimism and convex capacities. Nonlinear sensitivity towards probability also is an important factor underlying insurance. In [47] it is found that people's common aversion to incomplete insurance cannot be explained by curvature of utility but can be explained by nonlinear probabilities.

This paper combines the preceding two developments. That is, we assume utils as outcomes and identify the books that can be made[1] against the rank-dependent models. It is easily seen that books cannot be made whenever the gambles considered are "comonotonic" (same ordering of events according to favorableness of outcomes). Examples will demonstrate that books must be due to hedging, optimism, and other phenomena that all deal with noncomonotonic gambles. To the degree that such phenomena are descriptively or even normatively desirable, the exclusion of books is unwarranted. This paper studies a comonotonic Dutch book-making principle that does not exclude books unless all acts are comonotonic and thereby does allow for hedging, optimism, ambiguity aversion, etc. We show that such a book-making principle is not only necessary, but also sufficient, for the rank-dependent models, given payment in utils. Hence, a new foundation of the rank-dependent models results.

As a by-product of our analysis we show that the book-making principle is closely related to an additivity condition for preferences that is well-known in decision theory and that has been extensively studied in the mathematics literature. In a mathematical sense, our result extends Yaari's [49] theorem from risk to uncertainty. It is remarkable that de Finetti's book-making principle, usually considered as inextricably associated with additive probabilities, can so easily be adapted to nonadditive

[1] making a book means a violation of the book-making principle

probabilities.

2 De Finetti's Dutch book-making Principle

This section analyzes de Finetti's book-making principle. $S = \{s_1, ..., s_n\}$ is a finite *state space*, with subsets called *events*. One of the states is true and the others are not true. A decision maker is uncertain about which state is true. *Outcomes* are real numbers designating money. A gamble is a state-contingent payoff, e.g. a financial asset. Formally, a *gamble* f is a function from the state space to the outcomes. Gamble f will generate outcome $f(s)$ if s is the true state of nature. Gambles are often identified with n-tuples and, hence, the set of gambles is identified with $\mathbb{R}^n$. Sometimes probabilities of the states are given. Then the state space is a probability space and gambles are random variables. In general, probabilities need not be given.

By $\succcurlyeq$ we denote the preference relation of the decision maker over the gambles. It is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all gambles f, g) and transitive. The notation $\succ$ and $\sim$ is as usual. *Strict monotonicity* holds if $f \succ g$ whenever $f > g$ ($f > g$ means that $f(s) > g(s)$ for all states s). For a gamble f, a *fair price* is an outcome x such that $x \sim f$. As usual, outcomes are identified with constant gambles.

The Dutch book-making principle, also called coherence by de Finetti, is based on the idea that a number of good decisions, when taken together, should be good still. "Taken together" is interpreted as outcome-wise addition. A Dutch book, defined formally hereafter, consists of a number of preferences that, when taken together, yield a loss for each state of nature. Obviously, such a result is not good and therefore the *(Dutch) book-making principle* requires that no Dutch book exists.

DEFINITION 1 A *(Dutch) book* consists of a number of preferences

$$f^1 \succcurlyeq g^1$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$f^n \succcurlyeq g^n \quad \text{with}$$

$$\sum_{j=1}^{n} f^j(s) < \sum_{j=1}^{n} g^j(s) \text{ for all states } s. \quad \square$$

In words, if replacing g^j by f^j is good for each j, then the joint result of these replacements should not be a sure loss. Our presentation differs from de Finetti's in four respects. First, de Finetti also considers multiplication by positive scalars, where the final condition in Definition 1 is replaced by the condition that $\sum_{j=1}^{n} \lambda^j f^j(s) < \sum_{j=1}^{n} \lambda^j g^j(s)$ for some positive λ^js. We have dropped such scalar-multiplication because addition seems to be a more natural way of combining gambles than scalar-multiplication and because the required implications can be derived from addition alone. Second, de Finetti considers a game situation where an outside person can take the decision maker up on any of his preferences. We have formulated the condition in a single-person decision making context so as to avoid distortions due to strategic considerations (see de Finetti [11], footnote (a) in the 1964 translation) and the state of information of the outside person.

Third, as will be demonstrated in Theorem 2, the book-making principle is based on two principles, strict monotonicity and *additivity* ($f \succcurlyeq g$ implies $f + h \succcurlyeq g + h$ for all gambles f, g, h). In his discussions, de Finetti emphasized monotonicity but we, as many other authors, think that the essence of the book-making principle lies in additivity ([7] p. 359 second full paragraph, [36]). For moderate stakes, additivity seems a reasonable condition. The receipt of act h does not change the situation or needs of the decision maker much and, hence, it seems reasonable that the preference between f and g is not affected.

Fourth, de Finetti did not invoke a completeness requirement imposed on all gambles but instead he took an arbitrary set of gambles and their fair prices as the initial domain of preference. Because all linear combinations were incorporated also, his domain was a linear subspace on which, through the fair prices, a weak order was obtained. The extension of the following result to linear subspaces is omitted for simplicity of the presentation.

THEOREM 2 *The following three statements are equivalent for* $\succcurlyeq$ *on* $\mathbb{R}^n$.

(i) There exist probabilities $p_1, ..., p_n$ *such that preferences maximize expected value* $f \mapsto p_1 f(s_1) + ... + p_n f(s_n)$.

(ii) $\succcurlyeq$ *is a weak order, for each gamble there exists a fair price, and no Dutch book can be made.*

(iii) $\succcurlyeq$ *is a weak order, for each gamble there exists a fair price, and additivity and strict monotonicity are satisfied.*

□

We end this section with some comments on related mathematical results. There are many results similar to the equivalence of (i) and (iii) with continuity instead of the fair price condition and with an invariance condition for scalar multiplication (homotheticity) added ([5] Theorem 4.3.1, [28], [33], [35], [48]. Additivity of preference amounts to commutativity of an ordering and an addition operation which is extensively studied in the mathematics literature ([4] Chapter 15, [18], [24] Section

2.2.5). These results often consider more general state spaces and outcome spaces. Wakker [46, Theorem A2.1] presented a related result that did not use scalar multiplication either but used a stronger monotonicity condition plus continuity. Trockel [42] and Candeal & Induráin [8] present results without monotonicity for the preference relation or the representing linear functional. The additivity axiom was also central in early axiomatizations of case-based decision theory [19, 20]. The mathematics is related to invariance of preference with respect to a mixing operation ([16], [45]) which similarly leads to linear representations. In Theorem 2, we did not seek for maximal mathematical generality. The purpose of the theorem is to present de Finetti's book-making principle as an individual coherence condition while avoiding game-theoretic complications.

3 Hedging, Uncertainty Aversion, and Comonotonic Books

We present three examples of violations of the Dutch book-making principle. The first illustrates how Dutch books can help uncover irrationalities and is primarily of descriptive interest. The second example is based on hedging which was put forward by Yaari [49, p. 104] as a rationale for the rank-dependent models. The third example, the Ellsberg paradox, shows how aversion to unknown probabilities leads to a Dutch book, illustrating once more that additive probabilities cannot describe this paradox.

EXAMPLE 3 Consider gambles on a roulette wheel. There are 37 states of nature, corresponding to one of the numbers 0, ..., 36 being selected. A bet of \$1 on a single number yields a net profit of \$36 − \$1 = \$35 if the number shows up and −\$1 otherwise. A gambler may be indifferent between betting on each of the numbers but prefer any such bet to not betting. The resulting preferences constitute a Dutch book:

$$
\begin{aligned}
(35,-1,-1,\ldots,-1) &\succcurlyeq (0,\ldots,0)\\
(-1,35,-1,\ldots,-1) &\succcurlyeq (0,\ldots,0)\\
&.\\
&.\\
&.\\
(-1,-1,-1,\ldots,35) &\succcurlyeq (0,\ldots,0) \ \text{ but}\\
(-1,-1,-1,\ldots,-1) &< (0,\ldots,0). \quad \square
\end{aligned}
$$

EXAMPLE 4 [Hedging] Assume that a coin is tossed once and the state space is {heads, tails}. $(20,0)$ denotes the gamble yielding \$20 if heads and \$0 if tails. The other gambles are defined similarly and relate to the same toss of the coin. The following preferences are natural but generate a Dutch book.

$$
\begin{aligned}
(9,9) &\succcurlyeq (20,0)\\
(9,9) &\succcurlyeq (0,20) \ \text{ but}\\
(18,18) &< (20,20).
\end{aligned}
$$

The preferences in this example are traditionally explained by expected utility with concave utility. For moderate stakes, however, utility is close to linear and an alternative explanation for the observed risk aversion seems to be more plausible. Such an alternative explanation will be provided later, based on a nonlinear sensitivity towards chance. Note that, when the gambles $(20,0)$ and $(0,20)$ are taken together, one gamble serves as a hedge for the other. $\square$

EXAMPLE 5 [Ellsberg Example] Assume an urn K (known) containing red and black balls in equal proportions and an urn A (unknown or ambiguous) containing red and black balls in an unknown proportion. From each urn a ball will be drawn at random and its color will be inspected. The state space is $\{B_kB_a, B_kR_a, R_kB_a, R_kR_a\}$, where B_kB_a refers to a black ball from urn K and a black ball from urn A, and the other states are defined similarly. Act $(1,0,1,0)$ yieds \$1 if the ball from A has color black and yields nothing otherwise; other acts are defined similarly. The following preferences are commonly observed for $\epsilon = 0$.

$$(1,1,0,0) \succcurlyeq (1+\epsilon, 0, 1+\epsilon, 0)$$

$$(0,0,1,1) \succcurlyeq (0, 1+\epsilon, 0, 1+\epsilon) \quad \text{but}$$

$$(1,1,1,1) < (1+\epsilon, 1+\epsilon, 1+\epsilon, 1+\epsilon).$$

For $\epsilon > 0$ sufficiently small the preferences will still hold and a Dutch book results. The left acts provide a hedge for each other as in the preceding example because their combination replaces risk with certainty. This same hedging takes place when the right acts are combined, but in addition the uncertainty about the probabilities is removed [27]. It is well-known that these preferences cannot be explained by expected utility or any other model using additive probabilities. □

The examples have something in common. In each case, good and bad outcomes of the gambles in a summation neutralize each other. In the first example, the gambles added are to some degree substitutes for each other. The hope for a good outcome of one gamble loses some of its force if the gamble is added to another gamble that already provides a similar hope. In the second example, a complementarity effect takes place in the addition of the gambles $(20,0)$ and $(0,20)$. The aversive zero outcome of each is compensated by the large \$20 outcome of the other (hedging).

In the third example, adding $(1+\epsilon, 0, 1+\epsilon, 0)$ and $(0, 1+\epsilon, 0, 1+\epsilon)$ removes the uncertainty about the unknown probabilities of the outcomes.

In each example, variability of one gamble is tempered by counter-variability of the other one. The mentioned interaction effects do not arise when the gambles added are "comonotonic." A set of gambles is *comonotonic* if for each pair of elements f, g there do not exist states s, t such that $f(s) > f(t)$ and $g(s) < g(t)$.

We next introduce a generalization of the book-making principle suggested by the preceding considerations. A *comonotonic (Dutch) book* is a book as in Definition 1 with the extra restriction that the set of gambles considered $(\{f^1, \ldots, f^n, g^1, \ldots, g^n\})$ is comonotonic. The *comonotonic (Dutch) book-making principle* requires that no comonotonic Dutch book exists. Similarly, *comonotonic additivity* means that $f \succcurlyeq g$ implies $f + h \succcurlyeq g + h$ for all comonotonic gambles f, g, h.

We next define Choquet expected value, the model characterized by the comonotonic Dutch book-making principle. It is the rank-dependent model for decision under uncertainty, i.e. the context where no probabilities are given. Because payment is in utils, no utility function need to be defined; put in other words, utility is assumed to be linear. We therefore use the term Choquet expected value instead of Choquet expected utility. A *capacity* W is a function $W : 2^S \to [0,1]$ satisfying (a) $W(\emptyset) = 0$, (b) $W(S) = 1$, and (c) W is nondecreasing with respect to set inclusion. *Choquet expected value* holds if there exists a capacity W such that

$$f \mapsto \sum_{j=1}^{n} \pi_j f(s_j)$$

represents $\succcurlyeq$, where the *decision weights* π_j are defined as follows. First, a permutation ρ is chosen such that $f(s_{\rho(1)}) \geqslant \cdots \geqslant f(s_{\rho(n)})$. Next, $\pi_{\rho(i)} = W(\{s_{\rho(1)}, \ldots, s_{\rho(i)}\} - W(\{s_{\rho(1)}, \ldots, s_{\rho(i-1)}\}$; in particular, $\pi_{\rho(1)} = W(s_{\rho(1)})$. The decision weights are nonnegative and sum to one.

THEOREM 6 *The following three statements are equivalent for the preference relation $\succcurlyeq$ on $I\!R^n$.*

(i) There exists a capacity W such that preferences maximize Choquet expected value.

(ii) The binary relation $\succcurlyeq$ is a weak order, for each gamble there exists a fair price, and no comonotonic Dutch book can be made.

(iii) The binary relation $\succcurlyeq$ is a weak order, for each gamble there exists a fair price, and comonotonic additivity and strict monotonicity are satisfied.

□

The risk seeking in Example 3 can be explained by a capacity W assigning a weight exceeding 1/36 to each number. This capacity implies an overweighting of unlikely events and risk seeking for long-shot options. In Example 4, hedging can be explained by a capacity W with $W(Heads) = W(Tails) < .45$. This choice yields a decision weight of less than .45 for the 20 outcome and a decision weight exceeding .55 for the zero outcome. Consequently, the observed risk aversion is not ascribed to diminishing marginal utility as this was traditionally done, but it is ascribed to the extra attention paid to the zero outcome. The aversion to unknown probabilities in Example 5 can be explained by any capacity W assigning a greater value to the events $\{B_kB_a, B_kR_a\}$ and $\{R_kB_a, R_kR_a\}$, describing the colors from the known urn K than to the events $\{B_kB_a, R_kB_a\}$ and $\{B_kR_a, R_kR_a\}$, describing the colors from the unknown urn A.

We end this section with some comments on related mathematical results. There have been several variations on Statement (iii) in the literature. [13] used comonotonic additivity together with continuity but without any monotonicity to characterize a nonmonotonic generalization of CEV. Already Schmeidler [37] used a comonotonic additivity condition for functionals, in combination with continuity, to characterize noncomonotonic CEV functionals; he also characterized the monotonic case. The latter case is also characterized in [22]. Schmeidler's [38] comonotonic mixture-invariance condition for preferences is famous. It was used to obtain linearity with respect to second-stage probabilities. Chateauneuf ([9], Theorem 1) generalized Schmeidler's preference condition, considering mixtures of outcomes rather than of probabilities.

4 Discussion

The book-making principle relies on linear utility. Linear utility is reasonable for moderate amounts of money ([14], [17], [25] p. 290, [26] p. 86, [32] p. 176, [34] p. 91). In fact, the rank-dependent model suggests that much of the deviations from expected value observed for moderate amounts of money, and traditionally ascribed to curvature of utility, is due to nonlinear sensitivity towards probability. This suggestion is supported empirically by Selten, Sadrieh, & Abbing [40]. They compared nonlinearity of outcome sensitivity with nonlinearity of sensitivity towards probability. For the small outcomes considered (ranging between $-\$1$ and $\$3$), nonlinear sensitivity towards probability was more pronounced.

Our model can be interpreted as a return to [30]. That paper, one of the earliest empirical studies of risk attitude, already used nonlinear probabilities rather than nonlinear utilities to explain the deviations from expected value. Yaari [49] also assumed linear utility in his derivation of rank-dependent utility for risk and our model can be considered the generalization of Yaari's model to uncertainty.

Many studies into the nature of nonadditive probabilities are going on today. If both utilities and probability weights are unknown, complex measurement methods have to be invoked ([1], [6], [21], [43]. We suggest that linear utility is a good approximation for moderate stakes and, hence, that gambles with moderate stakes provide an easy tool for measuring nonlinear probability weighting ([23]; Diecidue, Wakker, & Zeelenberg, in preparation).

Appendix: Proofs

Proof of Theorem 2. The implication (i) $\Rightarrow$ (ii) follows from substitution. Next we assume (ii) and derive (iii). For strict monotonicity, assume that $f \succcurlyeq g$ and $f(s) < g(s)$ for all s. The preferences immediately entail a Dutch book and, hence, a contradiction. Strict monotonicity follows. For each gamble f, define $FP(f)$ as the fair price of gamble f. FP is uniquely determined and represents preference ($f \succcurlyeq g$ if and only if $FP(f) \geqslant FP(g)$; note that $x > y$ implies x^y because of strict monotonicity). We claim that FP satisfies *additivity* ($FP(f+g) = FP(f) + FP(g)$, also known as Cauchy's functional equation). To wit, if $FP(f+g) < FP(f) + FP(g)$ then a Dutch book

$$
\begin{aligned}
f &\succcurlyeq FP(f) \\
g &\succcurlyeq FP(g) \\
FP(f+g) &\succcurlyeq f+g
\end{aligned}
$$

$$f + g + FP(f + g) < f + g + FP(f) + FP(g)$$

results and, hence, a contradiction. If $FP(f+g) > FP(f)+FP(g)$ then the reversed preferences result in a Dutch book. Additivity of FP follows. Additivity of FP implies additivity of $\succcurlyeq$; hence, Statement (iii) follows.

We finally assume (iii) and derive (i). FP is defined as above and represents preference. We again derive additivity of FP. $f \sim FP(f)$ implies, by two-fold application of additivity (with $\succcurlyeq$ and with $\preccurlyeq$), that $f + g \sim FP(f) + g$. Additivity and $g \sim FP(g)$ imply that $g + FP(f) \sim FP(g) + FP(f)$. Transitivity implies that $f + g \sim FP(f) + FP(g)$; hence, $FP(f + g) = FP(f) + FP(g)$.

Additivity means that Cauchy's functional equation holds which, together with strict monotonicity, implies that FP is a linear functional ([2] Theorem 2.1.1.1). $FP(f) = \sum_{j=1}^{n} p_j f(s_j)$ for real numbers p_j. The p_js are nonnegative for if one, say p_1, were negative then we could find a gamble $(M, 1, \ldots, 1)$ with M so large that the FP of the gamble would be negative, implying that it is less preferred than the 0 gamble, thus violating strict monotonicity. Finally, $FP(1) = 1$ implies that the p_js sum to one. Statement (i) has been proved. $\square$

Proof of Theorem 6. The implication (i) $\Rightarrow$ (ii) follows from substitution. The implication (ii) $\Rightarrow$ (iii) is established as in the proof of Theorem 2, with the appropriate comonoticity requirements added; note that constant gambles are comonotonic with all other gambles.

We finally assume (iii) and derive (i). That FP is representing and satisfies *comonotonic additivity* ($FP(f + g) = FP(f) + FP(g)$ holds whenever f and g are comonotonic) is demonstrated exactly as in the proof of Theorem 2, again with all appropriate comonotonicity requirements added. We finally show that FP is a Choquet integral.

For any event E and real λ, λE denotes λ times the indicator function of E. For any fixed E, $\lambda \mapsto FP(\lambda E)$ satisfies Cauchy's equation on the nonnegative reals. On that set, the mapping is bounded on a nondegenerate interval, i.e., it is bounded above on $[0,1]$ by $FP(2,\ldots,2)$. Hence, FP is linear on this set ([2], Theorem 2.1.1.1) and $FP(\lambda E) = \lambda W(E)$ for the real number $W(E) = FP(1E)$. $W(\emptyset) = 0$ and $W(S) = 1$ follow because FP assigns fair prices. W is monotonic with respect to set inclusion: If $A \supset B$ but $W(A) < W(B)$, then we can find λ sufficiently large to imply $FP(\lambda A + (1,\ldots,1)) = FP(\lambda A) + FP(1,\ldots,1)) < FP(\lambda B)$, contradicting strict monotonicity. Hence W is monotonic with respect to set inclusion, which implies that W is nonnegative.

Every gamble can be written as a sum $\sum_{j=1}^{n} \lambda_j E_j - (M,\ldots,M)$ for nonnegative λ_j, nonnegative M, and decreasing sets $E_1 \supset \cdots \supset E_n$. To wit, in E_n the gamble is minimal, its second-smallest value is taken in $E_{n-1}E_n$, etc.; if the minimal value is negative then M is taken positive so as to have λ_n nonnegative). . By comonotonic additivity, $FP(\sum_{j=1}^{n} \lambda_j E_j - (M,\ldots,M)) = \sum_{j=1}^{n} FP(\lambda_j E_j) - FP(M,\ldots,M)) = \sum_{j=1}^{n} \lambda_j W(E_j) - M$ which is the CEV value of the gamble with respect to the capacity W. $\square$

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